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Decentralised controller design for a large-scale linear discrete-time polytopic uncertain systems

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ABSTRACT
One of the great challenges to control theory is to design controllers for the increasing size of uncertain large-scale systems. Such problems arise in many real-word applications and the systems are too large and too complex to be centralised controlled. For these reasons, the complex system is split into several interconnected subsystems and it is controlled in a decentralized fashion. In this article, a novel, original approach to the control of large-scale uncertain linear discrete-time systems by a decentralised controller is presented. In the existing references, such systems are divided into two groups: complex systems with strong or weak interactions. However, the new proposed approach depends on whether the system is stable or not. The design procedure consists of two steps: In the first step, the dynamic properties of the closed-loop decentralised controlled subsystems are determined in such a way that the stability, robustness, and performance of the closed-loop subsystems and the entire system are guaranteed. In the second step, a decentralised control algorithm must be designed that ensures the required properties of the subsystems that have been obtained in the first step. The advantage of this approach is that the decentralized controller design is performed on the subsystem level.

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1. Introduction

Stability analysis of large-scale systems (LSS) is an essential topic in control system theory; see Siljak (1978) and the references therein. The notion of system complexity indicates the fundamental characteristics of a complex system, including multidimensionality, uncertainty, and the impossibility of being centralised controlled. Direct analysis of the stability of such systems is generally hampered by the complexity of the overall system. This is why the problems of analysis and controller synthesis for complex systems are divided into independent/almost independent subproblems and controlled in this way by an algorithm with information constraints-decentralised control. Decentralised controller design procedures have been developed since the 1970s, Bakule (2008), Y. H. Chen (1989), Davison and Chang (1987), Davison et al. (2020), Gielen and Lazar (2015), Ikeda et al. (1981), Siljak (1978) and Wang and Davison (1973). Analysing the stability of each smaller resulting subsystem separately, that is, by neglecting interconnections, is a tractable but highly conservative approach (Siljak, 1978).

In the last three decades, the control of LSS using decentralised controllers has been successful. Decentralised controller design procedures have been developed in the frequency and time domains. In the frequency domain, the most interesting results belong to independent design (Hovd & Skogestad, 1993a), sequential design (Hovd & Skogestad, 1993b), and the method of equivalent subsystem (Kozáková et al., 2019).

The decentralised controller design methods obtained in the time domain could be divided into the following three groups: The methods using the aggregation matrix approach, Siljak (1978), for linear and nonlinear systems, the vector Lyapunov function approach, Matrosov (1962), and great progress has been made using the LMI-BMI approach. The survey of decentralised controller design procedures for continuous-time and discrete-time systems may be found in the excellent survey (Bakule, 2008) and the

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book (Davison et al., 2020). Numerous decentralised design methods developed, to date, fall into the six groups mentioned above. The disadvantage of these methods is when designing a decentralised controller, it is usually necessary to use a full mathematical model of the complex system. Decentralised controller design procedures on the subsystem level for continuous-time system are described in Kozáková et al. (2019) and Veselý (2021) and for discrete-time systems in Kozáková et al. (2019). All decentralised controller design procedures belong to the class of 'highly conservative approach'. These methods provide the decentralised controller design procedure using the properties of a complex system, such as strong or weak interactions. The above properties do not play any role in this paper. Conditions for decentralised controllability, observability and the decentralised stabilisation of the complex systems are given in Naiqi et al. (1988) and the references therein. Decentralised control for discrete-time linear systems in the frequency domain is presented in Kozáková et al. (2019), and for the time domain in Gielen and Lazar (2015), Rosinová and Halická (1994), Xu et al. (2019) and Yan and Bitmead (1992) and the references therein.

This paper aims to derive a new approach to the design of a robust decentralised controller for linear uncertain discrete-time systems. The main advantage of the present method is that decentralised controllers are designed at the subsystem level without considering the interaction links. The presented method is implemented in two steps. In the first step, one determines whether a complex uncertain system without a controller is stable or not. This calculation determines the required properties of the subsystems that guarantee the stability, robustness, and dynamic properties of the subsystems as well as of the closed-loop complex system. In the second step, the parameters of the decentralised controller will be calculated at the subsystem level to guarantee the required dynamic properties of the subsystems obtained in the first step. The results obtained in this paper present a new principal way and theory (see Conclusion) of the design of decentralised controllers, which could be split into two directions:

(1) decentralised controller design for a stable complex system,
(2) decentralised controller design for an unstable complex system.

The idea of a two-step procedure was already outlined in Veselý (2021) for continuous-time systems. The present article elaborates on the previous one and presents a new necessary and sufficient condition for the decentralised controller for the complex system, then a new robust stability condition for the complex system, and also a new specification of the equivalent subsystem when designing a decentralised controller for an unstable complex system.

The organisation of the paper is as follows. Section 2 provides preliminary results and a formulation of the problem. Section 3 introduces an equivalent subsystem and its use to the design of decentralised controllers for linear uncertain discrete-time LSS. Section 4 presents two examples showing the effectiveness of the proposed method. In Conclusion, Section 5, the advantages of the proposed method are presented.

In the sequal, the following notation will be adopted. Given a symmetric matrix $P = P^T \in R^{n\times n}$, the inequality $P > 0$, $(P < 0)$ denotes the positive (negative) definiteness of the matrix. Furthermore, $I_n, 0_n$ denotes the identity, zero matrices of dimension n.

2. Preliminaries and problem formulation

2.1. System description

We are given a large-scale uncertain discrete-time invariant system of the form

$$\begin{aligned} x(t+1) &= A(\xi)x(t) + B(\xi)u(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^l$ are state, control input, and controlled output, respectively. The matrices

$$(A(\xi), B(\xi)) = \sum_{i=1}^{N} (A_i, B_i)\xi_i \tag{2}$$

belong to a polytopic uncertainty domain with N-vertices and constant or time varying-uncertainties that belong to the set $\xi(t) \in \Omega_\xi$

$$\Omega_\xi = \left\{ \xi_i(t) \geq 0,\ i = 1, 2, \dots, N,\ \sum_{i=1}^{N} \xi_i(t) = 1 \right\} \tag{3}$$

and

$$\sum_{i=1}^{N} (\xi_i(t) - \xi_i(t-1)) = 0.$$

The matrices A_i, B_i, and C are with constant entries. We assume that the matrices B_i, C, $i = 1, 2, \ldots, N$ have a decentralised structure, X. B. Chen and Stankovic (2005), namely

$$A_i = \begin{bmatrix} A_{i11} & \ldots & A_{i1m} \\ A_{i21} & \ldots & A_{i2m} \\ \vdots & \ddots & \vdots \\ A_{im1} & \ldots & A_{imm} \end{bmatrix} \in R^{n\times n},$$

$$B_i = blockdiag[B_{i1} \ldots B_{im}] \in R^{n\times m},$$

$$C = blockdiag[C_1 \ldots C_m] \in R^{l\times n}. \quad (4)$$

We further assume that the complex system (1) is centralised controllable, and observable, Gong and Aldeen (1992), Seraji (2007) and Wang and Davison (1973).

The system (1) can be formally decomposed into subsystems in different ways. In this article, the division of the above matrices into sub-matrices reflects the inherent properties of the complex system. In the following, we will use the non-overlapping structure (X. B. Chen & Stankovic, 2005).

2.2. Preliminaries and problem formulation

The system (1) is asymptotically stable if $A(\xi)$ is Schur stable for all ξ_i, $i = 1, 2, \ldots, N$, that is, the eigenvalues of $A(\xi)$ are smaller than one in modulus.

Lemma 2.1 readily follows the Lyapunov stability theory (Vassiliki et al., 1988).

Lemma 2.1: *The sum of two discrete-time matrices $G + H \in R^{n\times n}$ is Schur stable if and only if a positive definite Lyapunov matrix $P > 0$ exists, such that*

$$(G + H)^T P(G + H) - P < 0 \quad (5)$$

holds.

Proof: The linear discrete-time system $x(t + 1) = Lx(t)$ is asymptotically stable if and only if a positive definite matrix $P > 0$ exists such that the Lyapunov inequality $L^T PL - L < 0$ holds, Vassiliki et al. (1988). Putting $L = G + H$ proves the lemma. ■

Definition 2.2: Let $E = \{e_{ij}\}_{n\times n}$ be the structured perturbation matrix of the system (1) where

- $e_{ij} = 1$ if there is an interaction connection between the subsystems i and j,
- $e_{ij} = 0$ if there is no connection between the subsystems i and j.

To reflect a weaker or varying interaction between the subsystems i and j, one may put the value of $e_{ij} \in < 0,\ 1 >$.

Definition 2.3 (Siljak, 1978): A complex system (1) is connective stable if and only if it is asymptotically, for all possible entries $e_{i,j}$ of the matrix $E(e_{ij})$.

In this article, the problem is to design the decentralised controller for each non-overlapping jth, $j = 1, 2, \ldots, m$ subsystem such that the decentralised controller guarantees the closed-loop stability, robustness, and performance of each subsystem, as well as the connective stability of the complex system with the designer-defined performance. The controller can be, for instance, a PI or PID controller. The control algorithm for a PID controller is as follows:

$$u(t)_j = kp_j C_j x(t)_j + ki_j C_j \sum_{to}^{\infty} x(t)_j + kd_j C_j \delta x(t)_j, \quad (6)$$

where

$$j = 1, 2, \ldots, m; \quad \delta x(t)_j = x(t)_j - x(t - 1)_j.$$

3. Robust decentralised controller design

3.1. Main theoretical results

In this section, we will focus our attention on finding such dynamic properties of the block diagonal matrix $Ad(\xi)$ of the complex system that ensure robust stability and performance of all subsystems, as well as of the complex dynamic system. We introduce an auxiliary block diagonal matrix: equivalent subsystem. The equivalent subsystem serves in the second step of the decentralised controller design procedure to design a decentralised PI/PID (or any) controller that will ensure the robust stability and performance of the block diagonal matrix with the decentralised controller $Ad(\xi) + Bd(\xi) * DecController$ and of the entire complex system. Assume that the complex system (1) is convex with respect to ξ. Let us split the system to the ith vertex of the polytope $i = 1, 2, \ldots, N$ having the following form

$$x(t + 1) = (Ad_i + Am_i)x(t) + B_i u(t),$$

$$y(t) = Cx(t), \tag{7}$$

where $i = 1, 2, \ldots, N$; Ad_i is the block diagonal part of the matrix A_i with m diagonal submatrices and $Am_i = A_i - Ad_i$ is the off-diagonal part of the system (1).

Lemma 3.1 (Gahinet et al., 1996): *Consider the quadratic function*

$$f(v) = (vAd_i + Am_i)^T P_i (vAd_i + Am_i) - P_i$$

and assume that $f(v)$ is convex, that is,

$$\frac{\delta^2 f(v)}{\delta^2 v} \geq 0.$$

Then $f(v)$ is negative in the rectangle $v \in < 0,\ vb >$, if and only if it takes negative values at the corners, that is, if and only if $f(v) < 0$ for $v = 0$ or $v = vb$.

Theorem 3.2: *Let two discrete-time system matrices Ad_i, Am_i with constant entries and a constant v belonging to the interval $< 0,\ vb >$ be given. The sum $G_i = vAd_i + Am_i$ is Schur stable for any such v if and only if a positive definite matrix P_i exists such that the following inequality holds:*

$$(vAd_i + Am_i)^{\mathrm{T}} P_i (vAd_i + Am_i) - P_i < 0. \tag{8}$$

Proof: Since $f(v) = (vAd_i + Am_i)^{\mathrm{T}} P_i (vAd_i + Am_i) - P_i$ is convex in v by Lemma 3.1, then Lemma 3.1 implies that $f(v)$ is negative definite if and only if it is so at the corners, that is, for $v = 0$ and $v = vb$. For $v = 0$, we have from (8) that

$$f(0) = Am_i^T P_i Am_i - P_i < 0. \tag{9}$$

Condition (9) is a necessary and sufficient condition for the existence of a decentralised controller for the system given by the matrix $Ad_i + Am_i$. If matrix Am_i is asymptotically stable, then due to Lemma 3.1, there exists the solution to variable v. For $v = vb$, we have (8), which proves the theorem. ■

For an uncertain polytopic system (1) with N vertices, the robust stability condition of the complex system with matrices G_i, $i = 1, 2, \ldots, N$, Veselý and Rosinová (2013) is as follows.

Theorem 3.3 (Peaucelle et al., 2000; Veselý & Rosinová, 2013): *Let us have two discrete-time matrices Ad_i, $Am_i, i = 1, 2, \ldots, N$ with constant entries and a positive constant $v \geq 0$. The uncertain polytopic system (1) with matrices $G_i = vAd_i + Am_i \quad i = 1, 2, \ldots, N$ is asymptotically stable for some positive coefficient $v = vb > 0$ if and only if there exist matrices $H, U \in R^{n \times n}$ and positive definite matrices $P_i > 0, i = 1, 2, \ldots, N$ such that the following inequality holds for all $i = 1, 2, \ldots, N$*

$$\begin{bmatrix} -P_i + G_i^T H^T + HG_i & -H + G_i^T U \\ -H^T + U^T G_i & P_i - (U^T + U) \end{bmatrix} < 0. \tag{10}$$

Remark 3.1: Note that if scalar v covers all subsystems, the value $v = vb$ obtained by solving (8) or (10) may be conservative (worst case). To decrease the conservatives one needs to use instead of a scalar v the diagonal matrix v as follows $v = blockdiag\{I_1 v_1, I_2 v_2, \ldots, I_m v_m\}$.

From Theorems 3.2 and 3.3, the following Corollary readily follows.

Corollary 3.4: *Let a value of $v = vb > 0$ be obtained due to the solution of (8) or (10). The complex system is asymptotically stable for some $v \geq 0$ if $v \in < 0,\ vb >$ holds.*

The two theorems serve to determine a value of v to check the stability of the complex system (1). The inequality (10) belongs to the class of BMI. If the complex system is of high order the elimination lemma and the linearisation approach should be used to obtain an LMI formulation (Veselý et al., 2011). Based on the calculated value of $v = vb > 0$ from (8) or for a polytopic system from (10), there are two possible paths to design a decentralised controller.

- The first one: if $vb \geq 1$, the LSS is stable without decentralised controllers. For ensuring the stability, robustness and performance of the subsystems and the LSS, any decentralised controller may be designed on the subsystem level that satisfies the following condition:

$$\begin{aligned} &|\lambda_k(Ad(i,j) + B(i,j) * DecController_j)| \\ &\quad \leq |\lambda_k(Ad(i,j) * vb)|, \\ &i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, m, \\ &k = 1, 2, \ldots, d_s(i,j) \leq n, \end{aligned} \tag{11}$$

  that is, the absolute values of closed-loop (i,j) subsystems dominant eigenvalues are less than or equal to the absolute values of open-loop modified dominant subsystem eigenvalues. The number of dominant (i,j) subsystem eigenvalues $d_s(i,j)$ which are to satisfy (11), is determined for the specific decentralised controller structure and parameters experimentally, see examples, obviously $d_s(i,j) = 1, 2$. Following Siljak (1978), the stability of a complex linear or nonlinear system can be determined using an aggregation matrix approach. If the complex system is stable and the absolute values of the diagonal elements of the aggregation matrix are not increased by the decentralised controllers, the stability of the complex system will not be affected. Thus, the results obtained in this paper agree with the results achieved using the aggregation matrix approach.

- The second possibility, if $vb < 1$, the LSS is unstable.The closed-loop stability of LSS is ensured if the robust decentralised controller is designed in such a way that the following condition holds for the subsystems dominant eigenvalues:

$$\begin{aligned}&|\lambda_k(Ad(i,j) + B(i,j) * DecController_j)|\\&\quad \le |\lambda_k(vbAd(i,j))|,\\&\quad i = 1, 2, \dots, N,\ j = 1, 2, \dots, m,\\&\quad k = 1, 2, \dots, d_s(i,j).\end{aligned} \tag{12}$$

Theorem 3.5: *Assume an ideal case, when the following equality holds for the i th vertex:*

$$|\lambda_k(Ad_i + B_i * DecControllers)| = |\lambda_k(vbAd_i))|,$$
$$i = 1, 2, \dots, N,\ k = 1, 2, \dots, d_s(i).$$

Then the robust stability boundary of the complex system is given by the circle of radius r_{sb} as follows:

$$r_{sb} = vb\left(\min_i\left(\max_k |\lambda_k(Ad_i)|\right)\right) \tag{13}$$

Proof: The result of (13) is obtained from inequality (12) for the dominant eigenvalue. ■

Remark 3.2: Theorem 3.5 serves as a sufficient robust stability condition for the LSS and provides the basis for the definition of an equivalent subsystem. For a stable complex system the absolute value of all subsystems eigenvalues need to be less than or equal to r_{sb}.

Remark 3.3: When the positive coefficient vb is known the minimal value of the degree of stability α_d for all subsystems $j = 1, 2, \dots, m$ can be calculated as follows:

$$\alpha_d = 1 - vb^2.$$

The following condition should hold for a stable LSS

$$\Delta V_j(x) \le -\alpha_d V_j(x), \quad j = 1, 2, \dots, m,$$

where $V_j(x)$, $j = 1, 2, \dots, m$, is the Lyapunov function of the closed-loop jth subsystem.

3.2. Equivalent subsystems

Equivalent subsystems have been introduced for the unstable LSS, when $vb < 1$. The equivalent subsystems serve in the second step to design a decentralised PI/PID (or any) controller that will ensure the stability, robustness, and performance of a block diagonal matrix with decentralised controller, $Ad(\xi) + B(\xi) * DecController$ and the complex system. Equivalent subsystems transform the subsystems matrices $Ad(i,j)$ so that the dominant eigenvalues of the corresponding subsystems increase by a specified value such that the designed decentralised controller for equivalent subsystem will guarantee simultaneously the stability, robustness, and performance of the subsystems and the complex systems. One approach to define the equivalent subsystems is as follows (see examples for two other approaches):

$$Ade(i,j) = U * V * \text{inv}(U), \tag{14}$$

where

$$[U, V] = eig(Ad(i,j));$$
$$u_d(i,j) = \max eig(V), k(i,j) = u_d(i,j) * (vb - \pi)$$

and where π is a small tuning parameter, for the first step calculation $\pi = 0$. Let us increase the maximum values of the first $d_s(i)$ dominant eigenvalues for the matrix V as follows $\max eig(V) = 2 - k(i,j)$, which due to (16) increases the dominant eigenvalues of the equivalent subsystem. Increasing π, the degree of closed-loop stability for the equivalent subsystem $Ade(i,j)$ will also be increased, which ensures the closed-loop LSS stability, robustness, and performance, Siljak (1978) and Veselý (1993). The above design procedure should be repeated $d_s(i)$ times to increase for ith all subsystems $d_s(i)$ dominant eigenvalues in the matrix V.

Let us assume a static output feedback with gain matrix K that guarantees the stability of the closed-loop equivalent subsystems. Then, the following corollary to Theorem 3.5 holds.

Corollary 3.6: *The closed-loop LSS (complex system) is asymptotically stable if either of the following two conditions holds:*

- $Ade(i,j) + B(i,j)K_jC$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, m$ *in all vertices are Schur stable, that is, all closed-loop equivalent subsystems are robustly stable.*
-

$$|\lambda_k(Ad_i + B_iKC)| \le |\lambda_k(vb * Ad_i)|, \quad i = 1, 2, \ldots, N, \ k = 1, 2, \ldots, d_s(i) \le n \tag{15}$$

holds for all closed-loop subsystems' dominant eigenvalues.

If the stability conditions do not hold, increase the value of the tuning parameter π and design a new decentralised controller (Siljak, 1978; Veselý, 1993). If the value of π approaches one and stability of the equivalent subsystems and LSS is still not guaranteed, change the structure of the decentralised controller, and/or use another decentralised controller design procedure (the authors recommend the regional pole placement approach).

4. Examples

The aim of the two examples presented below is to illustrate the advantages of the theory and the efficiency of calculation of robust decentralised controllers compared to known methods from the literature.

4.1. First example

This example covers several real processes. A complex uncertain system of order 7 consists of three subsystems having orders 2, 3, and 2. The complex uncertain polytopic system has two vertices ($N = 2$). For a system with real parameters, see Appendix.

Auxiliary parameters are $ro * I \ge P$ (*Lyapmatrix*), $ro = 25$, and the tuning matrix $Q = 0.4 * I$. The goal is to design a robust PI decentralised controller. The design procedure will be based on a simple condition that ensures the stability of the closed-loop system: the first difference in parameter-dependent Lyapunov function needs to be negative definite (semidefinite). This can be summarised as follows.

Let equivalent subsystems $Ade(i,j)$, $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, m$ be given. To obtain a decentralised robust PI controller let us assume that there are two auxiliary matrices $N_1(j), N_2(j) \in R^{n_j \times n_j}$ and a positive definite Lyapunov matrix $P(i,j)$ exist such that the following inequality holds for the first difference in Lyapunov function:

$$\begin{aligned}
\Delta V(t) &= x(t)^T W(i,j) x(t) < 0,\\
W(i,j) &= \{w(i,j)\}_{2\times 2}\},\\
w11(i,j) &= P(i,j) + N_1(j) + N_1(j)^T,\\
w12(i,j) &= -N_1(j)^T Ace(i,j) + N_2(j),\\
w21(i,j) &= w12(i,j)^T,\\
w22(i,j) &= -N_2(j)^T Ace(i,j) - Ace(i,j)^T N_2(j)\\
&\quad + Q(j) - P(i,j),\\
Ace(i,j) &= Ade(i,j) + B(i,j)K(j),\\
K(j) &= [(Kp(j) + Ki(j))C(j) \ \ Ki(j)].
\end{aligned}$$

The following two cases are presented.

Case a. For the calculation of LSS stability boundary suppose that only one parameter v covers all three subsystems. The tuning parameter $\pi = 0.2$. Using BMI script (10), one obtains for $vb = 1.2279$ that the complex system is stable without decentralised controllers. Due to Corollary 3.4, note that if decentralised controllers are designed to satisfy $v \in < 0, vb >$, the closed-loop complex system will be stable. The equivalent subsystems for all $i = 1, 2$ and $j = 1, 2, 3$ are calculated using (10) and three robust PI decentralised controllers are obtained as

$$u(t)_j = kp_jC_jx(t)_j + ki_jC_j\sum_{to}^{\infty} x(t)_j,$$

where

$$\begin{aligned}
R1 &: kp_1 = -0.5772, & ki_1 &= -1.0825,\\
R2 &: kp_2 = -1.8649, & ki_2 &= -1.3502,\\
R3 &: kp_3 = -0.6122, & ki_3 &= -1.1332,
\end{aligned}$$

or in symbolic way

$$R_1 = -0.5772 - \frac{1.0825}{s}, \quad R_2 = -1.8649 - \frac{1.3502}{s},$$

$$R_3 = -0.6122 - \frac{1.1332}{s}.$$

The absolute values of the closed-loop eigenvalues for the complex system are for $i = 1$: $EigLSS_1 = \{0.9413, 0.9029, 0.9087, 0.9087, 0.1593, 0.4881, 0.4881, 0.4131, 0.4131, 0.496\}$, for $i = 2$: $EigLSS_2 = \{0.9434, 0.9434, 0.903, 0.903, 0.1873, 0.4458, 0.4458, 0.4027, 0.4027, 0.5351\}$. The close-loop complex system is stable.

Case b. For the same system as above, put the tuning parameter $\pi = 0$. The obtained value of $vb = 1.2279$ is the same as in *Case a*. However, the obtained decentralised controllers do not guarantee the stability of LSS. The closed-loop system is unstable, but it is very close to the stability boundary, the maximal value of the closed-loop eigenvalues being as follows: $EigLSS1 = \{1.0221, 1.0221, \dots, 0.1956\}$, $EigLSS2 = \{1.0081, 1.0081, \dots, 0.2054\}$. The above two cases illustrate the role of the tuning parameter π in the decentralised controller design procedure.

4.2. Second example

This example illustrates the design of a robust PID decentralised controller using the method of regional pole placement, Gahinet et al. (1996), Peaucelle et al. (2000), Rosinová et al. (2021), and Yang et al. (2022). Following Peaucelle et al. (2000), the D_R region for closed-loop subsystem is defined as

$$D_R = \{z \in C : R_{11} + R_{12}z + R_{12}^T z^* + R_{22}zz^* < 0\} \tag{16}$$

If the all closed-loop subsystem eigenvalues lie in the prescribed D_R region, the closed-loop matrix $Adc(\xi)$ is said to be D_R stable. The closed-loop system $Adc(\xi)$ is D_R stable if and only if there exists a positive definite matrix $P(\xi)$ such that

$$R_{11} \otimes P(\xi) + R_{12} \otimes P(\xi)Adc(\xi) + R_{12}^T \otimes Adc(\xi)^T + R_{22} \otimes Adc(\xi)^T P(\xi) Adc(\xi) < 0$$

where $\otimes$ stands for the Kronecker product. Let us recall (Rosinová et al., 2021) that the extended first Lyapunov function difference for the D_R region is as follows:

$$\Delta V_{\text{ext}} = v^T \begin{bmatrix} R_{11} \otimes P(\xi) & R_{12} \otimes P(\xi) \\ (*) & R_{22} \otimes P() \end{bmatrix} v \le 0,$$

where $v^T = [(1_d \otimes x(t)^T)\ (1_d \otimes x(t+1)^T)]$.

The extended first difference in the Lyapunov function will be used to design a robust decentralised controller. The PID controller algorithm is given as in (6). Using the augmented state as a new state of the controller dynamics (Rosinová et al., 2021), a PID control algorithm will be obtained in the following form

$$u(t) = [kp + ki + kd \quad kd \quad ki - kd]y(t) = Ky(t).$$

See Rosinová et al. (2021) for more details. The complex system is given by (1) as an LSS system of order four with two subsystems of order 2 each, and with a polytopic uncertainty having two vertices, $N = 2$. The constraints of the Lyapunov matrix $P \le 500 * I$. The system matrices are

$i = 1$

$$A_1 = \begin{bmatrix} 0.5 & 0.8 & 0.1 & 0.15 \\ 0.1 & 0.3 & 0.25 & 0.15 \\ 0.11 & 0.22 & 0.6 & 0.65 \\ 0.08 & 0.13 & 0.1432 & 0.4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$i = 2$

$$A_2 = \begin{bmatrix} 0.38 & 0.55 & 0.095 & 0.11 \\ 0.12 & 0.22 & 0.22 & 0.13 \\ 0.09 & 0.15 & 0.55 & 0.58 \\ 0.08 & 0.11 & 0.161 & 0.38 \end{bmatrix},$$

$$B_2 = B_1, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

At the first step, the value of vb is calculated. We obtain $vb = 0.8723$, which indicates that the LSS is unstable. The D_R region (16) is chosen as a circle with centre at the origin and radius rd and the following three cases are considered. In the first case, equivalent subsystems are calculated by (14). For such equivalent subsystems, the LSS is stable for the D_R region with radius $rd \in \langle 0.55 - 0.71 \rangle$. In the second case, the equivalent subsystems are defined as follows $Ade(i,j) = Ad(i,j)(1/(vb - \pi))$, where $\pi = 0$ is the tuning parameter, and for the prescribed D_R region, the LSS is stable for radius $rd \in \langle 0.48 - 0.91 \rangle$. In the third case, the equivalent subsystems are chosen as follows:

$$Ade(i,j) = Ad(i,j) * blockdiag(\text{inv}[v - \pi_o \quad v - \pi_o$$

$$\nu - \pi_1 \quad \nu - \pi_1]); \quad \pi_o = 0,\ \pi_1 = 0.3.$$

Now, we will proceed with the second case of equivalent subsystems. The goal is to design two PID robust decentralised controllers such that the closed-loop equivalent subsystems' eigenvalues lay in the prescribed D_R region. In what follows, we will consider two cases of the D_R region with different radii $rd = 0.5$ and $rd = 0.95$. For instance, consider the extended state subsystem model of the first subsystem with a PID controller as in Rosinová et al. (2021)

$$Ad(1,1) = \begin{bmatrix} 0.5 & 0.8 & 0 & 0 \\ 0.1 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$Eig(Ad(1,1)) = \{0, 1, 0.7, 0.1\}.$$

The eigenvalues 1, 0 are due to the I, D part of controller. This simple example shows that the sufficient LSS robust stability condition (15) may not be satisfied for all eigenvalues. Stability boundary of the complex system is $r_{sb} = 1.vb = 0.8723$, (13). For $rd = 0.5$, using the approach described in Rosinová et al. (2021), one obtains the following decentralised controller parameters.

For the first subsystem:

$$R1 = -0.1932 - \frac{1.7542}{s} - 0.028\ \mathrm{s}$$

and for the second one:

$$R2 = -0.2822 - \frac{1.7912}{s} - 0.0509\ \mathrm{s}.$$

The absolute values of the closed-loop subsystems eigenvalues for the nominal subsystem model and two vertices $Eigclosednom_{ij}$, $i = 1, 2$, $j = 1, 2$ are

for the first vertex, $i = 1$

$$Eigclosednom_{11} = \{0.415, 0.3409, 0.3409, 0.174\},$$
$$Eigclosednom_{12} = \{0.5673, 0.348, 0.348, 0.2961\},$$

and for the second vertex, $i = 2$

$$Eigclosednom_{21} = \{0.65, 0.3113, 0.3113, 0.00973\},$$
$$Eigclosednom_{22} = \{0.6637, 0.3342, 0.3342, 0.2608\}.$$

Note that the D_R region is prescribed for both equivalent subsystems. For the obtained robust decentralised controllers, the LSS with two vertices is stable, and the absolute values of the closed-loop eigenvalues are as follows.

$i = 1$

$$EigclosedLSS_1 = \{0.6111, 0.5960, 0.5960, 0.2759, 0.3048, 0.3048, 0.271, 0.271\},$$

$i = 2$

$$EigcloseLSS_2 = \{0.7045, 0.5052, 0.4803, 0.4803, 0.3561, 0.3561, 0.1029, 0.2512\}.$$

Based on (15), the stability test of the LSS is determined for the case of $i = 1$ and two subsystems.

$$\begin{bmatrix} 1 \\ 0.7 \\ 0.1 \\ 0 \\ 1 \\ 0.82 \\ 0.1789 \\ 0 \end{bmatrix} 0.8723 \geq \begin{bmatrix} 0.415 \\ 0.3409 \\ 0.3409 \\ 0.174 \\ 0.5673 \\ 0.348 \\ 0.348 \\ 0.2961 \end{bmatrix} \begin{bmatrix} yes \\ yes \\ no \\ no \\ yes \\ yes \\ no \\ no \end{bmatrix}.$$

Obtained from the above inequality, the value of $d_s(i,j)$, (15) is for two subsystems $d_s(i) = 2 + 2$, that is for two subsystems, all four dominant eigenvalues satisfy inequality (15), guaranteeing in this way the robust stability of the complex system.

Another experiment shows that the robust stability condition (15) is satisfied for $rd = 0.84$. In this case of $i = 1, 2$ and two subsystems the condition $d_s(i,j) > 0$ holds and the LSS is stable as can be seen from the closed-loop LSS eigenvalues.

$$EigclosedLSS1 = \{0.8664, \ldots, 0.2717\},$$
$$EigclosedLSS2 = \{0.9206, \ldots, 0.2625\}.$$

Another interesting experiment is to consider $rd = 0.91$. For $i = 1$, 2 and two subsystems, the stability conditions (15) are as follows:

$$i = 1, \quad \begin{bmatrix} 1 \\ 0.7 \\ 0.1 \\ 0 \\ 1 \\ 0.82 \\ 0.1789 \\ 0 \end{bmatrix} 0.8723 \geq \begin{bmatrix} 0.8354 \\ 0.5932 \\ 0.5932 \\ 0.2148 \\ 0.9139 \\ 0.5936 \\ 0.5936 \\ 0.2952 \end{bmatrix} \begin{bmatrix} yes \\ yes \\ no \\ no \\ no \\ yes \\ no \\ no \end{bmatrix}$$

$$i=2,\quad \begin{bmatrix}1\\0.5691\\0.0309\\0\\1\\0.7822\\0.1478\\0\end{bmatrix} 0.8723 \geq \begin{bmatrix}0.9414\\0.5525\\0.5525\\0.1612\\0.9624\\0.5856\\0.5856\\0.2736\end{bmatrix}\begin{bmatrix}no\\no\\no\\no\\no\\yes\\no\\no\end{bmatrix}.$$

For the radius of $rd = 0.91$, the eigenvalues of LSS at the two vertices indicate that the LSS is stable but very near to the stability boundary. The condition (15) indicates the same problem, that is, $d_s(2,1) = 0$, but $d_s(1,1) = 2$. The closed-loop eigenvalues of the LSS are

$$EigclosedLSS_1 = \{0.9359, \ldots, 0.2744\},$$
$$EigclosedLSS_2 = \{0.9902, \ldots, 0.2645\}.$$

We have done still another experiment, when the equivalent subsystems are given as follows.

$$Ade(i,j) = Ad(i,j) * blockdiag(\text{inv}[v - \pi_o \quad v - \pi_o \quad v - \pi_1 \quad v - \pi_1]); \quad \pi_o = 0, \quad \pi_1 = 0.3.$$

The radius of the D_R region is still $rd = 0.91$. The stability requirement determines the following values of the coefficients $d_s(i,j)$ as $d_s(1,1) = 2, d_s(1,2) = 2, d_s(2,1) = 1, d_s(2,2) = 2$, that is, the complex system is stable and the closed-loop eigenvalues are

$$EigLSS1 = \{0.7807, \ldots, 0.2891\},$$
$$EigLSS2 = \{0.8405, \ldots, 0.2278\}.$$

When comparing the above two results for the same radius, it is clear that best results are obtained with the third equivalent subsystem.

Now, we take the radius of D_R region $rd = 0.95$. The absolute values of the closed-loop LSS eigenvalues for the two vertices $i = 1, 2$ are

$$EigclosedLSS_1 = \{0.9758, 0.836, \ldots, 0.2757\},$$
$$EigclosedLSS_2 = \{1.03, 0.9397, \ldots, 0.2657\}.$$

This indicates that the LSS is unstable. Stability test (15) for $i = 1, 2$ and two subsystems gives

$$i=1,\quad \begin{bmatrix}1\\0.7\\0.1\\0\\1\\0.82\\0.1789\\0\end{bmatrix} 0.8723 \geq \begin{bmatrix}0.876\\0.5926\\0.5926\\0.2169\\0.9535\\0.5946\\0.5946\\0.2968\end{bmatrix}\begin{bmatrix}no\\yes\\no\\no\\no\\yes\\no\\no\end{bmatrix},$$

$$i=2,\quad \begin{bmatrix}1\\0.5691\\0.0309\\0\\1\\0.7822\\0.1478\\0\end{bmatrix} 0.8723 \geq \begin{bmatrix}0.9826\\0.5527\\0.5527\\0.1629\\1.03\\0.5866\\0.5866\\0.2755\end{bmatrix}\begin{bmatrix}no\\no\\no\\no\\no\\yes\\no\\no\end{bmatrix}.$$

Stability condition (15) is not satisfied because for $i = 2$ and for the first subsystem, it holds $d_s(2,1) = 0$, while the others $d_s(i,j) = 1$. Thus, the closed-loop LSS is not stable, see Remark 3.1.

Summarising the above results to a table from the view of robust stability of the LSS, we obtain Table 1.

Table 1 shows that the robust stability boundary condition $r_{sb} = 0.8723$ is satisfied for all radii including $rd = 0.95$. This indicates that the robust stability boundary condition could be used with the sufficient condition as the stability criterion for the LSS. This example also shows that the third method of defining the equivalent subsystems is superior. The above two examples show, if the number of subsystems grows only the first step controller design procedure computation complexity increases, the complexity of the decentralised controllers design procedure does not change.

Table 1. The view of robust stability of the LSS.

Uncert	$i = 1$	$i = 1$	$i = 2$	$i = 2$	Result
Subs	1	2	1	2	–
rd	$d_s(1,1)$	$d_s(1,2)$	$d_s(2,1)$	$d_s(2,2)$	–
0.5	2	2	2	2	Stable
0.84	2	1	1	1	Stable
0.88	2	1	0	1	Stable
0.91	2	1	0	1	Stable
0.91	2	2	1	2	Stable3[a]
0.95	1	1	0	1	Unstable

[a]This refers to the third equivalent subsystem model.

5. Conclusion

This article presents a new method for designing a robust decentralised controller for a linear complex discrete-time system. The method consists of two steps. In the first step, dynamic properties of the subsystems are calculated that will ensure the stability, robustness and performance of the complex system. In the second step, decentralised controllers of the subsystems are designed to meet the requirements obtained in the first step. The main advantage of the present method is that decentralised controllers are designed at the subsystem level without the need to consider the interaction links.

The benefits of this paper can be summarised as follows:

(1) This paper fundamentally changes the view of a complex system. The complex system is not divided according to the size of the interaction links, whether the bonds are strong or weak, but according to whether the complex system is stable or not.
(2) For a stable complex system, the design procedure of a decentralised controller is significantly simpler and the design principle is that the dynamic properties of the subsystem with the proposed controller should not be deteriorated.
(3) For an unstable system, we have proposed to use an equivalent subsystem approach, which simplifies the decentralised controller design process. One of the best methods to design decentralised controllers is the regional pole placement.
(4) We have obtained a necessary and sufficient condition for a decentralised controller (9).
(5) The conservatism of the presented method depends on the used methods of decentralised controller design in the second step, see Remark 3.1.
(6) We have obtained the new theoretical results about the stability of complex system stability calculation for discrete-time complex systems.
(7) Inequality (10) belongs to the class of BMI. If the complex system is of a high order, the elimination lemma and the linearisation approach should be used to obtain an LMI formulation. Efficient solvers exist in the literature.
(8) Some questions about the stability of a complex system are left for further research, especially a better interpretation of stability conditions (12) and the choice of the equivalent subsystem structure, see Second Example.

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Data availability statement

All data are available within the article. Data are original and have not been published yet.

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References

Bakule, L. (2008). Decentralized control: An overview. *Annual Reviews in Control*, *32*(1), 87–98. https://doi.org/10.1016/j.arcontrol.2008.03.004

Chen, X. B., & Stankovic, S. S. (2005). Decomposition and decentralized control of systems with multi-overlaping structure. *Automatica*, *41*(10), 1765–1772. https://doi.org/10.1016/j.automatica.2005.01.020

Chen, Y. H. (1989). Large-scale uncertain systems under insufficient decentralized controller. *Journal of Dynamic Systems, Measurement, and Control*, *111*(3), 359–363. https://doi.org/10.1115/1.3153061

Davison, E. J., Anghdam, A. G., & Miler, D. E. (2020). *Decentralized control of large-scale systems*. Springer-Verlag US.

Davison, E. J., & Chang, T. N. (1987). Decentralized stabilization and pole assignment for general proper systems. *IEEE Transactions on Automatic Control*, *35*(6), 652–664. https://doi.org/10.1109/9.53544

Gahinet, P., Apkarian, P., & Chilali, M. (1996). Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *IEEE Transactions on Automatic Control*, *41*(3), 436–442. https://doi.org/10.1109/9.486646

Gielen, R. H., & Lazar, M. (2015, May). On stability analysis methods for large-scale discrete-time systems. *Automatica*, *55*, 66–72. https://doi.org/10.1016/j.automatica.2015.02.034.

Gong, Z., & Aldeen, M. (1992). On the characterization of fixed modes in decentralized control. *IEEE Transactions on Automatic Control*, *37*(7), 1046–1050. https://doi.org/10.1109/9.148369

Hovd, M., & Skogestad, S. (1993a). Improved independent design of robust decentralized controllers. *Journal of Process Control*, *3*(1), 43–51. https://doi.org/10.1016/0959-1524(93)80014-3

Hovd, M., & Skogestad, S. (1993b). Sequential design of decentralized controllers. *Automatica*, *30*(10), 1601–1607. https://doi.org/10.1016/0005-1098(94)90099-X

Ikeda, M. I., Siljak, D. D., & White, D. E. (1981). Decentralized control with overlapping information sets. *Journal of Optimization Theory and Applications*, *34*(2), 279–310. https://doi.org/10.1007/BF00935477

Kozáková, A., Veselý, V., & Kučera, V. (2019). Robust decentralized controller design based on equivalent subsystems. *Automatica*, *107*(3), 29–35. https://doi.org/10.1016/j.automatica.2019.05.031

Matrosov, V. M. (1962). On the theory of stability motion. *Journal of Applied Mathematics and Mechanics*, *26*(6), 1506–1522. https://doi.org/10.1016/0021-8928(62)90189-2

Naiqi, W., Renhou, L., & Baosheng, H. (1988, August). Decentralized controllability and observability, and decentralized stabilization. In *Proceedings of the 1988 IEEE International Conference on Systems, Man, and Cybernetics* (pp. 598–601). IEEE. https://doi.org/10.1109/ICSMC.1988

Peaucelle, D., Alzelier, D., Bachelier, O., & Bernussou, J. (2000). A new robust D-stability condition for real convex polytopic uncertainty. *Systems and Control Letters*, *40*(1), 21–30. https://doi.org/10.1016/S0167-6911(99)00119-X

Rosinová, D., & Halická, M. (1994, September). Decentralized stabilization of discrete-time systems: Subsystem robustness approach. In J. Mikles & M. Huba (Eds.), *IFAC new trends in design of control systems* (pp. 209–213). Elsevier.

Rosinová, D., Veselý, V., & Hypiusová, M. (2021, October). Novel robust gain scheduled PID controller design using D_R regions. *Asian Journal of Control*, 1–12. https://doi.org/10.1002/asjc.2633

Seraji, H. (2007). On fixed modes in decentralized control systems. *International Journal of Control*, *35*(5), 775–784. https://doi.org/10.1080/00207178208922654

Siljak, D. D. (1978). *Large-scale dynamic systems: Stability and structure*. North Holland.

Vassiliki, M., Hennet, J. C., & Bitsoris, G. (1988). Feedback control of linear discrete-time systems under state and control constraints. *International Journal of Control*, *47*(6), 1727–1735. https://doi.org/10.1080/00207178808906132

Veselý, V. (1993). Large scale dynamic system stabilization using the principle of dominant subsystem approach. *Kybernetika*, *29*(1), 48–61.

Veselý, V. (2021). Novel approach to decentralized controller design for large scale uncertain linear systems. *International Journal of Innovative Computing, Information and Control*, *17*(5), 1571–1580. https://doi.org/10.24507/ijicic.17.05.1571

Veselý, V., & Rosinová, D. (2013). Robust PID-PSD controller design: BMI approach. *Asian Journal of Control*, *15*(2), 469–478. https://doi.org/10.1002/asjc.2013.15.issue-2

Veselý, V., Rosinová, D., & Kučera, V. (2011). Robust static output feedback controller LMI based design via Elimination. *Journal of the Franklin Institute*, *348*(9), 2468–2479. https://doi.org/10.1016/j.jfranklin.2011.04.020

Wang, S. H., & Davison, E. (1973). On the stabilization of decentralized control systems. *IEEE Transactions on Automatic Control*, *18*(5), 473–478. https://doi.org/10.1109/TAC.1973.1100362

Xu, J., Xu, L., Xie, L., & Zhang, H. (2019, May). Decentralized control for linear systems with multiple input channels. *Science China Information Sciences*, *62*(5), 52202. https://doi.org/10.1007/s11432-018-9617-0

Yan, W. Y., & Bitmead, R. R. (1992). Control of linear discrete-time periodic systems: A decentralized control approach. *IEEE Transactions on Automatic Control*, *37*(10), 1644–1648. https://doi.org/10.1109/9.256403

Yang, R., Rotondo, D., & Puing, V. (2022). LMI-based design of state-feedback controllers for pole clustering LPV systems in a union of D_R-regions. *International Journal of Systems Science*, *53*(2), 291–312. https://doi.org/10.1080/00207721.2021.1954717

Appendix. Parameters of complex system from Section 4.1

Assume that uncertainties ξ_i, $i = 1, 2, \ldots, N$ are constants.

First vertex, $i = 1$

$$A1 = \begin{bmatrix} 0.22 & 0.7 & 0.0057 & 0.0175 & 0.05 & 0.0275 & 0.0075 \\ 0.1 & 0.25 & 0.045 & 0.0125 & 0.025 & 0.0175 & 0.01 \\ 0.067 & 0.01 & 0.12 & 0.37 & 0.28 & 0.0375 & 0.0092 \\ 0.003 & 0.0123 & 0.4 & 0.04 & 0.1 & 0.05 & 0.025 \\ 0.0175 & 0.01 & 0.3 & 0.41 & 0.12 & 0.0375 & 0.01 \\ 0.005 & 0.0088 & 0.02 & 0.0107 & 0.03 & 0.35 & 0.31 \\ 0.0175 & 0.01 & 0.05 & 0.03 & 0.0027 & 0.412 & 0.35 \end{bmatrix}.$$

Second vertex, $i = 2$

$$A2 = \begin{bmatrix} 0.21 & 0.65 & 0.0032 & 0.0175 & 0.03 & 0.0025 & 0.0075 \\ 0.1 & 0.33 & 0.03 & 0.0075 & 0.02 & 0.0175 & 0.01 \\ 0.0055 & 0.0103 & 0.161 & 0.35 & 0.23 & 0.0375 & 0.0067 \\ 0.003 & 0.0123 & 0.35 & 0.13 & 0.05 & 0.03 & 0.025 \\ 0.0175 & 0.0088 & 0.02 & 0.0083 & 0.003 & 0.25 & 0.1523 \\ 0.0187 & 0.0103 & 0.005 & 0.0255 & 0.0027 & 0.25 & 0.34 \end{bmatrix}.$$

Input and output matrices

$$B1 = B2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C1 = C2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$